Exercise 1. 1. It follows by a direct algebraic computation.

2. By the properties of the cross product, we have $\langle \partial_{x_i}\vec{\Phi}, \partial_{x_1}\vec{\Phi}\times\partial_{x_2}\vec{\Phi}\rangle = 0$ for all $i = 1,2$, which implies that $\langle \partial_{x_i}\vec{\Phi}, \vec{n}\rangle = 0$ for all $i = 1,2$. This implies that

$$\begin{aligned}\langle \Delta_g\vec{\Phi}, \vec{n}\rangle &= \frac{1}{\sqrt{\det(g)}}\sum_{i,j=1}^{n} g^{i,j}\left\langle \partial_{x_i}\left(\sqrt{\det(g)}\partial_{x_j}\vec{\Phi}\right), \vec{n}\right\rangle \\ &= \frac{1}{\sqrt{\det(g)}}\sum_{i,j=1}^{2} g^{i,j}\left(\partial_{x_j}\left(\sqrt{\det(g)}\right)\left\langle \partial_{x_j}\vec{\Phi}, \vec{n}\right\rangle + \sqrt{\det(g)}\left\langle \partial^2_{x_i,x_j}\vec{\Phi}, \vec{n}\right\rangle\right) \\ &= \sum_{i,j=1}^{2} g^{i,j}\left\langle \partial^2_{x_i,x_j}\vec{\Phi}, \vec{n}\right\rangle.\end{aligned}$$

The formula now follows from the fact that

$$\begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{1,2} & g_{2,2}\end{pmatrix} = \frac{1}{g_{1,1}g_{2,2}-g_{1,2}^2}\begin{pmatrix} g_{2,2} & -g_{1,2} \\ -g_{1,2} & g_{1,1}\end{pmatrix}.$$

3. We have

$$\partial_{x_i}\vec{\Phi}_t = \partial_{x_i}\vec{\Phi} + t\,\partial_{x_i}v\,\vec{n} + t\,v\,\partial_{x_i}\vec{n},$$

and since $|\vec{n}| = 1$, we have $\partial_{x_i}\vec{n} = 0$, which shows that $\partial_{x_i}\vec{n} = \lambda_1^i\,\partial_{x_1}\vec{\Phi} + \lambda_2^i\,\partial_{x_2}\vec{\Phi}$, and we have

$$\begin{aligned} -\mathbb{I}_{i,1} &= \langle \partial_{x_i}\vec{n}, \partial_{x_1}\vec{\Phi}\rangle = g_{1,1}\lambda_1^i + g_{1,2}\lambda_2^i \\ -\mathbb{I}_{i,2} &= \langle \partial_{x_i}\vec{n}, \partial_{x_2}\vec{\Phi}\rangle = g_{1,2}\lambda_1^i + g_{2,2}\lambda_2^i,\end{aligned}$$

which implies that

$$\begin{pmatrix}\lambda_1^i \\ \lambda_2^i\end{pmatrix} = -\frac{1}{g_{1,1}g_{2,2}-g_{1,2}^2}\begin{pmatrix} g_{2,2} & -g_{1,2} \\ -g_{1,2} & g_{1,1}\end{pmatrix}\begin{pmatrix}\mathbb{I}_{1,i} \\ \mathbb{I}_{2,i}\end{pmatrix} = \begin{pmatrix} -g^{1,1}\mathbb{I}_{1,i} - g^{1,2}\mathbb{I}_{2,i} \\ -g^{1,2}\mathbb{I}_{1,i} - g^{2,2}\mathbb{I}_{2,i}\end{pmatrix}.$$

Notice however that this step is not necessary Therefore, we get

$$\begin{aligned}&\partial_{x_1}\vec{\Phi}_t\times\partial_{x_2}\vec{\Phi}_t = \left(\partial_{x_1}\vec{\Phi} + t\,\partial_{x_1}v\,\vec{n} + t\,v\,\partial_{x_1}\vec{n}\right)\times\left(\partial_{x_2}\vec{\Phi} + t\,\partial_{x_2}v\,\vec{n} + t\,v\,\partial_{x_2}\vec{n}\right) \\ &= \partial_{x_1}\vec{\Phi}\times\partial_{x_2}\vec{\Phi} + t\left(\partial_{x_1}v\frac{1}{\det(g)}\left(-g_{2,2}\partial_{x_1}\vec{\Phi} + g_{1,2}\partial_{x_2}\vec{\Phi}\right) + \partial_{x_2}v\frac{1}{\det(g)}\left(-g_{1,1}\partial_{x_2}\vec{\Phi} + g_{1,2}\partial_{x_1}\vec{\Phi}\right)\right) \\ &- t\,v\,\det(g)\left(g^{1,1}\mathbb{I}_{1,1} + 2g^{1,2}\mathbb{I}_{1,2} + g^{2,2}\mathbb{I}_{2,2}\right)\vec{n} + O(t^2) \\ &= \sqrt{\det(g)}\,(1 - 2\,t\,H\,v)\,\vec{n} + t\,\mu_1\partial_{x_1}\vec{\Phi} + t\,\mu_2\partial_{x_2}\vec{\Phi} + O(t^2)\end{aligned}$$

where we used that

$$\begin{aligned}&(\partial_{x_1}\vec{\Phi}\times\partial_{x_2}\vec{\Phi})\times\partial_{x_1}\vec{\Phi} = \langle\partial_{x_1}\vec{\Phi}, \partial_{x_1}\vec{\Phi}\rangle\partial_{x_2}\vec{\Phi} - \langle\partial_{x_2}\vec{\Phi}, \partial_{x_1}\vec{\Phi}\rangle\partial_{x_1}\vec{\Phi} \\ &= g_{1,1}\partial_{x_2}\vec{\Phi} - g_{1,2}\partial_{x_2}\vec{\Phi} \\ &(\partial_{x_1}\vec{\Phi}\times\partial_{x_2}\vec{\Phi})\times\partial_{x_2}\vec{\Phi} = \langle\partial_{x_1}\vec{\Phi}, \partial_{x_2}\vec{\Phi}\rangle\partial_{x_2}\vec{\Phi} - \langle\partial_{x_2}\vec{\Phi}, \partial_{x_2}\vec{\Phi}\rangle\partial_{x_1}\vec{\Phi} \\ &= g_{1,2}\partial_{x_2}\vec{\Phi} - g_{2,2}\partial_{x_1}\vec{\Phi} \\ &\partial_{x_1}\vec{n}\times\partial_{x_2}\vec{\Phi} = -\sqrt{\det(g)}\left(g^{1,1}\mathbb{I}_{1,1} + g^{1,2}\mathbb{I}_{2,1}\right)\vec{n} \\ &\partial_{x_1}\vec{n}\times\partial_{x_2}\vec{n} = -\sqrt{\det(g)}\left(g^{1,2}\mathbb{I}_{1,2} + g^{2,2}\mathbb{I}_{2,2}\right)\vec{n}.\end{aligned}$$

Therefore, we deduce that

$$\mathrm{Area}(\vec{\Phi}_t) = \mathrm{Area}(\vec{\Phi}) - 2t\int_{\Omega} H\, v\,\sqrt{\det(g)}dx + O(t^2),$$

which shows that $\vec{\Phi}$ is critical for the area if and only if $H = 0$ identically.

Exercise 3. A direct computation shows that

$$\begin{aligned} &E = 1 + (f'(z))^2, \quad F = 0, \quad G = |f(z)|^2 \\ &L = \frac{f}{|f|}\frac{-f''(z)}{\sqrt{1+|f'(z)|^2}}, \quad M = 0, \quad N = \frac{|f(z)|}{\sqrt{1+|f'(z)|^2}}, \end{aligned}$$

which shows that $H = 0$ if and only if

$$f(z)f''(z) = 1 + |f'(z)|^2.$$

We rewrite the ODE as

$$\frac{f'(z)f''(z)}{1+|f'(z)|^2} = \frac{f'(z)}{f(z)}.$$

Integrating, we get

$$\frac{1}{2}\log(1+|f'(z)|^2) = \log(f(z)) + \log(C),$$

or

$$-\log(C) = \log\left(\frac{f(z)}{\sqrt{1+|f'(z)|^2}}\right),$$

so the function $\dfrac{f(z)}{\sqrt{1+|f'(z)|^2}}$ is constant. Recalling that $\dfrac{d}{dt}\mathrm{arccosh}(t) = \dfrac{1}{\sqrt{t^2-1}}$, we easily conclude the proof of the theorem.